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Stochastic functional evolution equations with monotone nonlinearity: Existence and stability of the mild solutions

Ruhollah Jahanipur

Department of Mathematics, Faculty of Science, University of Kashan, 87317-51167, Ravand Road, Kashan, Iran

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ABSTRACT

In this paper, we study a class of semilinear functional evolution equations in which the nonlinearity is demicontinuous and satisfies a semimonotone condition. We prove the existence, uniqueness and exponentially asymptotic stability of the mild solutions. Our approach is to apply a convenient version of Burkholder inequality for convolution integrals and an iteration method based on the existence and measurability results for the functional integral equations in Hilbert spaces. An Itô-type inequality is the main tool to study the uniqueness, p-th moment and almost sure sample path asymptotic stability of the mild solutions. We also give some examples to illustrate the applications of the theorems and meanwhile we compare the results obtained in this paper with some others appeared in the literature.

1. Introduction

Because of its wide application in various sciences such as physics, mechanical engineering, control theory and economics, the theory of stochastic partial (functional) differential equations has been attended by many authors over the last years. Two of the most important problems in this theory are the existence of some kind of solution [9,22] and the (asymptotic) stability of the solutions [19,21]. Adopting the framework of Gelfand triple, some authors examined the two problems for strong solutions by the variational method under the coercivity and monotone conditions and some others took the tools of semigroup theory which provides a unified approach to investigate in the properties of the mild solutions for a wide variety of stochastic partial differential equations. For example, Haussman [11] and Ichikawa [12,13] obtained the sufficient conditions for the existence, stability in the mean-square sense and of a.s. sample paths of strong and mild solutions via Lyapunov second

E-mail address: jahanipu@kashanu.ac.ir.

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method. Following the ideas of [11–13], Caraballo [3] and Caraballo and Real [4] extended the same results to the strong solutions of stochastic delay evolution equations. Caraballo, Liu and Truman [6] and also Caraballo, Garrido-Atienza and Real [7] studied the existence, uniqueness and stability in the second moment of the strong solutions to a class of fully nonlinear stochastic partial functional equations, first in the finite dimensions and then in infinite dimensional Hilbert spaces by the method of Galerkin's approximation. Using the energy equality, Taniguchi [27] proved the stability in the second moment of the energy solutions to nonlinear stochastic partial functional equations with finite delay in Hilbert space setting. The case of stochastic functional differential equations with infinite delay has been also tried both for the existence and for the stability of the strong and mild solutions [10].

Although, the construction of a Lyapunov function for a special problem in hand is in general very difficult, but while we are concerned with the strong solutions, the Lyapunov second method is a powerful technique to prove the stability results. On the other hand, the mild solutions do not necessarily have stochastic differentials and one cannot apply the Itô's formula to them. Therefore, the technique seems to be inconvenient for investigating the stability of the mild solutions. To overcome this difficulty, Ichikawa [12,13] and K. Liu [20] introduced approximating systems with strong solutions and then used a limiting argument to transfer the stability properties to the mild solutions. On the other hand, Taniguchi [25,26] and Jahanipur [14–16] solved the problem by giving an L^p-estimate for the difference of two mild solutions and proved the stability of the mild solutions of semilinear stochastic (delay) evolution equations. However, the results in [14,16] are more general than the previous works in that the nonlinear part of the equation is of monotone-type and satisfies the weak condition of demicontinuity.

In this paper, we will generalize the results of [14,16] to semilinear stochastic functional evolution equations. Our novelty is that the nonlinear part of the equation is demicontinuous and satisfies a special kind of monotone condition instead of Lipschitz one. In addition to this, the linear part involves a time-dependent family of unbounded linear operators generating an exponentially stable evolution operator. Since there are not any standard results on the existence of the mild solutions for the class of evolution equations to which the present paper is about, we devote Section 4 to accomplish this task. The proofs of the existence and uniqueness are based on the Itô-type inequality [14] and a version of Burkholder inequality which is established in Section 3. In Section 5, we study the asymptotic behaviour of the mild solutions and finally in Section 6, the results are applied to general stochastic functional initial-boundary value problems.

2. Preliminaries

Let H be a real separable Hilbert space with the norm and inner product denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Let T be a positive real number. A family

$$\big\{U(t,s)\colon\ 0 \leqslant s \leqslant t \leqslant T\big\}$$

of bounded linear operators on H is said to be an *evolution operator* if

- $U(t,t) = I$, $U(t,r)U(r,s) = U(t,s)$, for $0 \leqslant s \leqslant r \leqslant t \leqslant T$, where I is the identity operator;
- the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant T$.

Let $\{A(t)\colon\ 0 \leqslant t \leqslant T\}$ be a family of closed densely defined linear operators on H whose domain D is independent of $t \in [0,T]$. We say that $U(t,s)$ is a strong evolution operator with generator $A(t)$ if the following hold:

(a) For all $s \leqslant t$ and for each $x \in D$,

$$U(t,s)x - x = \int_s^t U(t,r)A(r)x\,dr.$$

(b) Let $x \in D$ and $s \in [0, T]$. For all $t > s$, we have $U(t, s)D \subseteq D$ and

$$\int_s^t A(r)U(r, s)x\,dr = \big(U(t, s) - I\big)x.$$

If $\{A(t)\colon\ 0 \leqslant t \leqslant T\}$ is the generator of a strong evolution operator $U(t, s)$ and if $x \in D$, then the function $t \mapsto U(t, s)x$ is differentiable on $s \leqslant t \leqslant T$ and satisfies $\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x$. Moreover, $s \mapsto U(t, s)x$ is differentiable on $[0, t]$ and

$$\frac{\partial}{\partial s}U(t, s)x = -U(t, s)A(s)x. \tag{2.1}$$

The following are the relevant hypotheses concerning A and U.

Hypothesis 2.1. There exists a $\lambda \in \mathbb{R}$ such that

(a) for each $t \in [0, T]$, $A(t) - \lambda I$ is the generator of a strongly continuous contraction semigroup; that is, $A(t) - \lambda I$ is a closed maximal monotone operator with dense domain;
(b) for each $\mu > \lambda$, the operator-valued function $(-A(t) + \mu I)^{-1}$ is strongly continuously differentiable with respect to $t \in [0, T]$;
(c) $B(t, s) = A(t)[\mu I - A(s)]^{-1}$ is uniformly bounded in (t, s) for $\mu > \lambda$ with a bound depending on μ.

It is turned out (see [1,2,18,23]) that under the above conditions, the family $U(t, s)$ is a strong evolution operator with generator $A(t)$ which is *exponentially bounded with parameter* λ on $[0, T]$; i.e., $\|U(t, s)\| \leqslant e^{\lambda(t-s)}$ for all $0 \leqslant s \leqslant t \leqslant T$. These conditions apply to a large class of parabolic, hyperbolic and functional evolution equations (see e.g. [8]).

Let K be another real separable Hilbert space. We use the same notations $\|\,.\,\|$ and $\langle .\,, .\rangle$ for the norm and inner product in K as well as in H. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete stochastic basis with a right continuous filtration.

Definition 2.2. A family of random linear functionals $\{W_t\colon\ t \geqslant 0\}$ on K is called a cylindrical Brownian motion on K, if

(a) $W_0 = 0$ and $W_t(x)$ is $\mathcal{F}_t$-adapted for every $x \in K$;
(b) for every $x \in K$ such that $x \neq 0$, $W_t(x)/\|x\|$ is a one-dimensional Brownian motion.

For the properties of cylindrical Brownian motion and its relation to other definitions of Brownian motion in infinite dimensions, see [28].

Definition 2.3. Let $\xi : [0, \infty) \to K$ be an $\mathcal{F}_t$-adapted, predictable process such that

$$E\left[\int_0^t \big\|\xi(s)\big\|^2\,ds\right] < \infty$$

for all $t \geqslant 0$. The stochastic integral of ξ with respect to the cylindrical Brownian motion $\{W_t\colon\ t \geqslant 0\}$ is a real-valued continuous martingale given by

$$\int_0^t \big\langle \xi(s), dW_s\big\rangle = \sum_{n=1}^{\infty} \int_0^t \big\langle \xi(s), e_n\big\rangle\,dW_s(e_n),$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis of K.

Assume that $L_2(K,H)$ is the space of Hilbert–Schmidt operators from K to H with the familiar Hilbert–Schmidt norm $\|\,.\,\|_2$. Now, we define the H-valued stochastic integral for $L_2(K,H)$-valued processes.

Definition 2.4. Let $\Phi : [0,\infty) \to L_2(K,H)$ be an $\mathcal{F}_t$-adapted, predictable process such that

$$E\left[\int_0^t \|\Phi(s)\|_2^2\,ds\right] < \infty$$

for all $t \geqslant 0$. The stochastic integral of Φ is an H-valued continuous martingale given by

$$\left\langle h, \int_0^t \Phi(s)\,dW_s \right\rangle = \int_0^t \langle \Phi^*(s)h, dW_s\rangle, \quad \forall h \in H,$$

where Φ^* is the adjoint operator of Φ.

For a fixed real $r > 0$, let $C_H = C(-r,0;H)$ be the Banach space of all continuous H-valued functions $\psi : [-r,0] \to H$ defined on the finite delay interval $[-r,0]$ with the usual sup-norm $\|\psi\|_{C_H} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|$. Also, whenever a problem of measurability is in concern, we will equip C_H with the σ-algebra of its Borel sets. Given any $p \geqslant 2$, denote by $C^p_{\mathcal{F}_0}$ the space of all continuous processes $\phi : [-r,0] \times \Omega \to H$ such that $\phi(\theta,.)$ is $\mathcal{F}_0$-measurable for each $\theta \in [-r,0]$ and $E(\sup_{-r\leqslant\theta\leqslant 0} \|\phi(\theta)\|^p) < \infty$. To any adapted continuous stochastic process

$$X : [-r,T] \times \Omega \to H \quad \text{such that} \quad E\big(\sup_{-r\leqslant t\leqslant T} \|X(t)\|^p\big) < \infty$$

(with $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-r,0]$) there corresponds a C_H-valued adapted process $X_t \in C^p_{\mathcal{F}_0}$ defined on $\theta \in [-r,0]$ by

$$X_t(\theta) = X(t+\theta), \quad t \in [0,T].$$

Consider on H a semilinear stochastic functional evolution equation of the form

$$\begin{cases} dX(t) = \big[A(t)X(t) + f(t,X_t)\big]\,dt + g(t,X_t)\,dW_t, & t \in [0,T], \\ X(\theta) = \phi(\theta), & \theta \in [-r,0], \end{cases} \tag{2.2}$$

where the initial data $\phi \in C^p_{\mathcal{F}_0}$.

Definition 2.5. An H-valued, $\mathcal{F}_t$-adapted predictable process $X(t)$, $t \in [-r,T]$, is called a mild solution of (2.2) if $\int_{-r}^T \|X(t)\|^2\,dt < \infty$ w.p. 1, and $X(t)$ satisfies the integral equation

$$X(t) = U(t,0)\phi(0) + \int_0^t U(t,s)f(s,X_s)\,ds + \int_0^t U(t,s)g(s,X_s)\,dW_s, \quad t \in [0,T], \tag{2.3}$$

with $X(\theta) = \phi(\theta)$ for $\theta \in [-r,0]$.

The main aim of this paper is to prove the existence, uniqueness and asymptotic stability of the mild solutions of (2.2) under the following hypotheses on A, U, g and the nonlinear part f.

Hypothesis 2.6.

(a) The function $f : [0, T] \times \Omega \times C_H \to H$ is jointly measurable.
(b) For each $t \in [0, T]$ and $\omega \in \Omega$, the mapping $\psi \mapsto f(t, \omega, \psi)$ is demicontinuous; i.e., whenever $\{\psi_n\}$ is a sequence which is strongly convergent to ψ in C_H, then $f(t, \omega, \psi_n)$ converges weakly to $f(t, \omega, \psi)$ in H.
(c) There exists a nonnegative number M such that for each $\omega \in \Omega$, the function $(t, \psi) \mapsto f(t, \omega, \psi)$ is semimonotone with parameter M. By this, we mean that

$$\big\langle f(t, \omega, \psi_1) - f(t, \omega, \psi_2), \psi_1(0) - \psi_2(0)\big\rangle \leqslant M\big\|\psi_1(0) - \psi_2(0)\big\|^2,$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C_H$.
(d) There exists a constant $C > 0$ such that

$$\big\|f(t, \omega, \psi)\big\| \leqslant C\big(1 + \|\psi\|_{C_H}\big),$$

for all $t \in [0, T]$, $\omega \in \Omega$ and $\psi \in C_H$.
(e) $g : [0, T] \times \Omega \times C_H \to L_2(K, H)$ is a predictable process on H such that

$$\big\|g(t, \omega, \psi_1) - g(t, \omega, \psi_2)\big\|_2 \leqslant C\|\psi_1 - \psi_2\|_{C_H},$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C_H$.
(f) A and U satisfy Hypothesis 2.1.

Finally, we give the Itô-type inequality [14] which is our main tool in this paper to prove the uniqueness, asymptotic stability and p-th mean boundedness of the mild solutions. Let $\{W_t\colon t \geqslant 0\}$ be the cylindrical Brownian motion with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Assume that $p \geqslant 2$ and f, g are two processes defined on $[0, T]$ with values in H and $L_2(K, H)$, respectively and satisfy

$$\int_0^T E\big\|f(t)\big\|^p\, dt < \infty, \qquad \int_0^T E\big\|g(t)\big\|_2^p\, dt < \infty.$$

Theorem 2.7 *(Itô-type inequality). Let ξ be an H-valued, $\mathcal{F}_0$-measurable random variable. Suppose that U and A satisfy Hypothesis* 2.1*. If*

$$X(t) = U(t, 0)\xi + \int_0^t U(t, s)f(s)\, ds + \int_0^t U(t, s)g(s)\, dW_s, \quad t \in [0, T],$$

then

$$\begin{aligned}\big\|X(t)\big\|^p \leqslant{}& e^{p\lambda t}\|\xi\|^p + p\int_0^t e^{p\lambda(t-s)}\big\|X(s)\big\|^{p-2}\big\langle X(s), f(s)\big\rangle\, ds \\ &+ p\int_0^t e^{p\lambda(t-s)}\big\|X(s)\big\|^{p-2}\big\langle X(s), g(s)\, dW_s\big\rangle \\ &+ \frac{p(p-1)}{2}\int_0^t e^{p\lambda(t-s)}\big\|X(s)\big\|^{p-2}\big\|g(s)\big\|_2^2\, ds,\end{aligned}$$

for all $t \in [0, T]$.

3. A Burkholder-type inequality

Zangeneh [29] proved a Burkholder-type inequality for the convolution integrals with respect to the martingales based on a version of the energy inequality. In this section, we give a proof in which the Itô-type inequality of Section 2 is used so that the result is more convenient for our purposes and contains more accurate bounds. First, we recall a lemma [29].

Lemma 3.1. *Let $X(t)$, $t \geqslant 0$, be an H-valued continuous process. If $M(t)$ is an H-valued continuous martingale, then for any constant $K > 0$ we have*

$$E\left(\sup_{0\leqslant\rho\leqslant t}\left|\int_0^{\rho}\langle X(s), dM(s)\rangle\right|\right) \leqslant \frac{3}{2K}E\big(X^*(t)\big)^2 + \frac{3K}{2}E\big([M](t)\big),$$

where $[\,]$ stands for the quadratic variation process and $X^*(t) = \sup_{0\leqslant s\leqslant t}\|X(s)\|$.

Theorem 3.2. *Let A and U satisfy Hypothesis 2.1 with $\lambda = 0$. Then for all $p \geqslant 2$ we have*

$$E\left(\sup_{0\leqslant\rho\leqslant t}\left\|\int_0^{\rho}U(\rho,s)g(s)\,dW_s\right\|^p\right) \leqslant e^{\gamma t}\frac{4C_p}{p}\int_0^T E\big\|g(s)\big\|_2^p\,ds,$$

where $\gamma = 2C_p(1-\frac{2}{p})$ and $C_p = \frac{p(p-1)}{2} + \frac{(3p)^2}{2}$.

Proof. Define for any positive integer n the map

$$R_n(t): H \to D, \qquad R_n(t) = n\big(nI - A(t)\big)^{-1}.$$

Then $R_n(t)$ is defined on all of H and since $\langle A(t)x, x\rangle \leqslant 0$, by Cauchy–Schwarz inequality, we have for all $x \in D$ that $\|x\| \leqslant \|n^{-1}(nI - A(t))x\|$. Therefore,

$$\big\|R_n(t)\big\| \leqslant 1 \quad \text{for } t \in [0, T].$$

Let $g_n(t) = R_n(t)g(t)$ and set

$$X(t) = \int_0^t U(t,s)g(s)\,dW_s, \qquad X_n(t) = \int_0^t U(t,s)g_n(s)\,dW_s.$$

By the Itô-type inequality (Theorem 2.7), we have

$$\begin{aligned}\big\|X_n(t)\big\|^p \leqslant{}& p\sup_{0\leqslant\rho\leqslant t}\left|\int_0^{\rho}\big\|X(s)\big\|^{p-2}\big\langle X_n(s), g_n(s)\,dW_s\big\rangle\right| \\ &+ \frac{p(p-1)}{2}\int_0^t\big\|X_n(s)\big\|^{p-2}\big\|g(s)\big\|_2^2\,ds,\end{aligned} \tag{3.1}$$

since $\|g_n(s)\|_2 \leqslant \|g(s)\|_2$. Taking the mathematical expectation of both sides of (3.1) and using Lemma 3.1, we obtain

$$E\|X_n(t)\|^p \leqslant \frac{3p}{2K}E\big(X_n^*(t)\big)^p + \left[\frac{p(p-1)}{2} + \frac{3pK}{2}\right]\int_0^t E\|X_n(s)\|^{p-2}\|g(s)\|_2^2\,ds,$$

where $K > 0$ is an arbitrary constant. Now, choose $K = 3p$ and use the elementary inequality

$$u^{1-\alpha}v^{\alpha} \leqslant (1-\alpha)u + \alpha v,$$

which is true for $u, v \geqslant 0$ and all $0 \leqslant \alpha \leqslant 1$. Then we obtain

$$E\big(X_n^*(t)\big)^p \leqslant \frac{1}{2}E\big(X_n^*(t)\big)^p + C_p\left(1 - \frac{2}{p}\right)\int_0^t E\|X_n(s)\|^p\,ds + \frac{2C_p}{p}\int_0^t E\|g(s)\|_2^p\,ds, \tag{3.2}$$

where $C_p = \frac{p(p-1)}{2} + \frac{(3p)^2}{2}$. At this point, we need to show that $E(X_n^*(t))^p < \infty$ to move the first term on the right of (3.2) to the left. By the stochastic version of integration by parts [9], we can write

$$X_n(t) = \int_0^t g_n(s)\,dW_s - \int_0^t \frac{\partial}{\partial s}\big(U(t,s)\big)\left(\int_0^s g_n(\tau)\,dW_\tau\right)ds.$$

Since $\int_0^s g_n(\tau)\,dW_\tau \in D$, we get by (2.1) that

$$\frac{\partial}{\partial s}\big(U(t,s)\big)\int_0^s g_n(\tau)\,dW_\tau = -U(t,s)A(s)\int_0^s g_n(\tau)\,dW_\tau, \quad s \in [0,t].$$

Therefore,

$$X_n(t) = \int_0^t g_n(s)\,dW_s + \int_0^t U(t,s)A(s)\left(\int_0^s g_n(\tau)\,dW_\tau\right)ds, \quad t \in [0,T]. \tag{3.3}$$

Now, consider the continuous martingale

$$M_n(t) = \int_0^t \big(I - A(0)\big)g_n(s)\,dW_s.$$

We can rewrite (3.3) as

$$X_n(t) = \int_0^t g_n(s)\,dW_s + \int_0^t U(t,s)A(s)\big(I - A(0)\big)^{-1}M_n(s)\,ds.$$

According to Hypothesis 2.1(c), there exists a constant $K > 0$ such that $\|A(s)(I - A(0))^{-1}\| \leqslant K$. Since $\|U(t,s)\| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$, we conclude that

$$E\big\|X_n(t)\big\|^p \leqslant 2^p E\left[\sup_{0\leqslant\rho\leqslant t}\left\|\int_0^\rho g_n(s)\,dW_s\right\|^p\right] + 2^p K^p T^{p-1}\int_0^t E\big\|M_n(s)\big\|^p\,ds$$
$$\leqslant 2^p c_p T^{\frac{p}{2}-1}\int_0^T E\big\|g(s)\big\|_2^p\,ds + 2^p K^p T^{p-1}\int_0^T E\big\|M_n(s)\big\|^p\,ds.$$

Here, we have used the Burkholder–Davis–Gundy inequality with the corresponding constant c_p and the fact that $\|g_n(s)\|_2 \leqslant \|g(s)\|_2$. But, Hypothesis 2.1(c) again implies the existence of a constant $K_n > 0$ with the property $\|(I - A(0))R_n(t)\| \leqslant K_n$ for all $t \in [0,T]$. Thus,

$$\int_0^T E\big\|M_n(s)\big\|^p\,ds \leqslant c_p K_n^p T^{\frac{p}{2}}\int_0^T E\big\|g(s)\big\|_2^p\,ds < \infty.$$

Therefore, $E(X_n^*(t))^p < \infty$ and applying this in (3.2) yields

$$E\big(X_n^*(t)\big)^p \leqslant 2C_p\left(1 - \frac{2}{p}\right)\int_0^t E\big(X_n^*(s)\big)^p\,ds + \frac{4C_p}{p}\int_0^T E\big\|g(s)\big\|_2^p\,ds.$$

Next, from the well-known Gronwall inequality, we see that

$$E\big(X_n^*(t)\big)^p \leqslant e^{\gamma t}\frac{4C_p}{p}\int_0^T E\big\|g(s)\big\|_2^p\,ds, \tag{3.4}$$

in which $\gamma = 2C_p(1 - \frac{2}{p})$. Finally, by Doob's inequality for convolution integrals (see [9]), we have

$$E\left(\sup_{0\leqslant t\leqslant T}\left\|\int_0^t U(t,s)\big(g_n(s) - g(s)\big)\,dW_s\right\|^2\right) \leqslant E\int_0^T \big\|g_n(s) - g(s)\big\|_2^2\,ds.$$

Since $R_n(s) \to I$ strongly and $\|R_n(s) - I\| \leqslant 2$, by Lebesgue's dominated convergence theorem, we get

$$E\int_0^T \big\|g_n(s) - g(s)\big\|_2^2\,ds \to 0, \quad \text{as } n \to \infty.$$

From this, one can conclude that $\sup_{0\leqslant t\leqslant T}\|X_n(t) - X(t)\| \to 0$ in L^2 and hence in probability. Then an application of Fatou's lemma in (3.4) completes the proof. □

4. Existence of a unique mild solution

Our aim in this section is to investigate the existence of a unique mild solution for (2.2) by means of the semigroup approach which gives a unified treatment of a wide class of deterministic and stochastic partial differential equations. Hence, the linear operator $A(t)$ contains derivatives with respect to the spatial variables (e.g. Laplacian operator) and the nonlinear part f depends for example on the values of the solution with time lag or delay. To prove the existence, we follow a successive approximation method and by virtue of the Burkholder-type inequality established in Section 3, we keep bounded the p-th mean of the approximations in each stage. Uniqueness is also a consequence of the Itô-type inequality (Theorem 2.7).

It is worth noting that a simple transformation helps us to reduce the problem to the case when $\lambda = 0$. Let $U_1(t,s) = e^{-\lambda(t-s)}U(t,s)$ and $A_1(t) = A(t) - \lambda I$. Then the family $\{A_1(t)\colon\ 0 \leqslant t \leqslant T\}$ generates the strong evolution operator $U_1(t,s)$ which is of contraction type; i.e., $\|U_1(t,s)\| \leqslant 1$. Hence, $\langle A_1(t)x, x\rangle \leqslant 0$ for all $x \in D$. Consider the function $\eta : [-r,T] \to \mathbb{R}$ which is defined as

$$\eta(t) = \begin{cases} e^{\lambda t}, & t \in [0,T], \\ 1, & t \in [-r,0]. \end{cases}$$

Assume also that $\tilde{f} : [0,T] \times \Omega \times C_H \to H$ is given by $\tilde{f}(t,\omega,\psi) = e^{-\lambda t} f(t,\omega,\eta_t\psi)$ and $\tilde{g} : [0,T] \times \Omega \times C_H \to L_2(K,H)$ is given by $\tilde{g}(t,\omega,\psi) = e^{-\lambda t} g(t,\omega,\eta_t\psi)$. Here η_t is the function defined on $[-r,0]$ by $\eta_t(s) = \eta(t+s)$. It is easy to see [17] that $\tilde{f}$ and $\tilde{g}$ satisfy the same conditions as for f, g in Hypothesis 2.6.

Proposition 4.1. *If $X(t)$ is the mild solution of* (2.2); *that is,*

$$X(t) = U(t,0)\phi(0) + \int_0^t U(t,s) f(s,X_s)\,ds + \int_0^t U(t,s) g(s,X_s)\,dW_s, \tag{4.1}$$

then

$$Y(t) = U_1(t,0)\phi(0) + \int_0^t U_1(t,s)\tilde{f}(s,Y_s)\,ds + \int_0^t U(t,s)\tilde{g}(s,Y_s)\,dW_s, \tag{4.2}$$

where $Y(t) = e^{-\lambda t}X(t)$ for $t \in [0,T]$ and $Y(\theta) = \phi(\theta)$ on $[-r,0]$. The converse is also true and so the two problems are equivalent.

Proof. Assume that (4.1) holds. Multiply both sides by $e^{-\lambda t}$ to obtain

$$e^{-\lambda t}X(t) = e^{-\lambda t}U(t,0)\phi(0) + \int_0^t e^{-\lambda(t-s)}U(t,s)e^{-\lambda s} f(s,X_s)\,ds$$
$$+ \int_0^t e^{-\lambda(t-s)}U(t,s)e^{-\lambda s} g(s,X_s)\,dW_s.$$

Then note that for all $t \in [0,T]$ one has

$$X_s(\theta) = \begin{cases} e^{\lambda(s+\theta)}Y(s+\theta) = (\eta_s Y_s)(\theta), & s+\theta \in [0,T], \\ \phi(s+\theta) = Y(s+\theta) = (\eta_s Y_s)(\theta), & s+\theta \in [-r,0]. \end{cases}$$

Any way, $e^{-\lambda s} f(s, X_s) = \tilde{f}(s, Y_s)$ and $e^{-\lambda s} g(s, X_s) = \tilde{g}(s, Y_s)$. Therefore, $Y(t)$ satisfies (4.2). A similar argument shows that the converse is also true and the proof is complete. □

Accordingly, we may assume in the remainder of this section that $\lambda = 0$. First of all, we are going to prove the uniqueness of a continuous adapted mild solution. To this end, we use the Itô-type inequality of Section 2. As before, let $p \geqslant 2$ and A and U satisfy Hypothesis 2.1.

Theorem 4.2. *The continuous adapted mild solution X of* (2.2) *with*

$$E\big(X^*(t)\big)^p < \infty \quad \text{for all } t \in [0, T], \tag{4.3}$$

if it exists, is unique.

Proof. Let $X(t)$ and $Y(t)$ be two adapted continuous mild solutions of (2.2) with the same initial data $\phi \in C^p_{\mathcal{F}_0}$ satisfying (4.3). Then

$$X(t) - Y(t) = \int_0^t U(t, s)\big(f(s, X_s) - f(s, Y_s)\big)\, ds + \int_0^t U(t, s)\big(g(s, X_s) - g(s, Y_s)\big)\, dW_s,$$

for all $t \in [0, T]$. Everything is ready for us to apply the Itô-type inequality with $p = 2$ which yields

$$\begin{aligned}\big\|X(t) - Y(t)\big\|^2 \leqslant 2 \int_0^t &\big\langle X(s) - Y(s), f(s, X_s) - f(s, Y_s)\big\rangle\, ds \\ &+ 2 \int_0^t \big\langle X(s) - Y(s), \big(g(s, X_s) - g(s, Y_s)\big)\, dW_s\big\rangle \\ &+ \int_0^t \big\|g(s, X_s) - g(s, Y_s)\big\|_2^2\, ds, \quad t \in [0, T].\end{aligned}$$

We estimate each term on the right side of the above inequality. By Hypothesis 2.6(c), we have for the first integral that

$$\int_0^t \big\langle X(s) - Y(s), f(s, X_s) - f(s, Y_s)\big\rangle\, ds \leqslant M \int_0^t \big\|X(s) - Y(s)\big\|^2\, ds. \tag{4.4}$$

Next, Lemma 3.1 with $K = 6$ implies that

$$\begin{aligned}&E\left(\sup_{0\leqslant \rho\leqslant t}\left|\int_0^\rho \big\langle X(s) - Y(s), \big(g(s, X_s) - g(s, Y_s)\big)\, dW_s\big\rangle\right|\right) \\ &\leqslant \frac{1}{4} E\Big(\sup_{0\leqslant s\leqslant t} \big\|X(s) - Y(s)\big\|^2\Big) + 9E\left(\int_0^t \big\|g(s, X_s) - g(s, Y_s)\big\|_2^2\, ds\right)\end{aligned}$$

$$\leqslant \frac{1}{4}E\Big(\sup_{0\leqslant s\leqslant t}\|X(s)-Y(s)\|^2\Big)+(3C)^2\int_0^t E\Big(\sup_{-r\leqslant\theta\leqslant 0}\|X(s+\theta)-Y(s+\theta)\|^2\Big)\,ds$$

$$\leqslant \frac{1}{4}E\Big(\sup_{0\leqslant s\leqslant t}\|X(s)-Y(s)\|^2\Big)+(3C)^2\int_0^t E\Big(\sup_{0\leqslant\theta\leqslant s}\|X(\theta)-Y(\theta)\|^2\Big)\,ds, \tag{4.5}$$

since $X=Y$ on $[-r,0]$. Finally, Hypothesis 2.6(e) gives us the following estimate for the expectation of the third integral:

$$\int_0^t E\|g(s,X_s)-g(s,Y_s)\|_2^2\,ds \leqslant C^2\int_0^t E\Big(\sup_{0\leqslant\theta\leqslant s}\|X(\theta)-Y(\theta)\|^2\Big)\,ds. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we conclude that

$$\frac{1}{2}E\Big(\sup_{0\leqslant s\leqslant t}\|X(s)-Y(s)\|^2\Big)\leqslant \bar{C}\int_0^t E\Big(\sup_{0\leqslant\theta\leqslant s}\|X(\theta)-Y(\theta)\|^2\Big)\,ds,$$

where $\bar{C}=2M+C^2+2(3C)^2$. Therefore, by the Gronwall inequality,

$$E\Big(\sup_{0\leqslant s\leqslant t}\|X(s)-Y(s)\|^2\Big)=0 \quad \text{for all } t\in[0,T].$$

Thus, $X(.,\omega)=Y(.,\omega)$ for almost all $\omega\in\Omega$ on $[0,T]$. $\square$

Now, we come to investigate the existence of a mild solution for (2.2).

Theorem 4.3. *Let $p\geqslant 2$. If $E(\sup_{0\leqslant s\leqslant t}\|g(s,0)\|^p)<\infty$ for all $t\in[0,T]$ and hypotheses 2.6 hold, then Eq. (2.2) has a unique continuous adapted mild solution X corresponding to the given initial data $\phi\in C^p_{\mathcal{F}_0}$ such that*

$$E\big(X^*(t)\big)^p<\infty, \quad \forall t\in[0,T].$$

Proof. Without lose of generality, we may let the initial data $\phi\equiv 0$. Indeed, define $\tilde{\phi}:[-r,T]\times\Omega\to H$ as

$$\tilde{\phi}(t,\omega)=\begin{cases}\phi(t,\omega), & t\in[-r,0],\\ U(t,0)\phi(0,\omega), & t\in[0,T].\end{cases}$$

Assume that X is a continuous solution of the integral equation (2.3) and set $\tilde{X}(t)=X(t)-\tilde{\phi}(t)$. Then $\tilde{X}$ is also continuous, $\tilde{X}_0=0$ and

$$\tilde{X}(t)=\int_0^t U(t,s)f(s,\tilde{X}_s+\tilde{\phi}_s)\,ds+\int_0^t U(t,s)g(s,\tilde{X}_s+\tilde{\phi}_s)\,dW_s.$$

Since the functions $\tilde{f} : [0, T] \times \Omega \times C_H \to H$ given by $\tilde{f}(t, \omega, \psi) = f(t, \omega, \psi + \tilde{\phi}_t)$ and $\tilde{g} : [0, T] \times \Omega \times C_H \to L_2(K, H)$ which is given by $\tilde{g}(t, \omega, \psi) = g(t, \omega, \psi + \tilde{\phi}_t)$ satisfy the same conditions as for f, g in Hypothesis 2.6, we can reduce solving equation (2.3) to that of the following:

$$\tilde{X}(t) = \int_0^t U(t,s)\tilde{f}(s, \tilde{X}_s)\,ds + \int_0^t U(t,s)\tilde{g}(s, \tilde{X}_s)\,dW_s.$$

Since $g(s, 0)$ is an $L_2(K, H)$-valued predictable process, then $\int_0^t g(s, 0)\,dW_s$ is an H-valued continuous martingale with the quadratic variation $\int_0^t \|g(s, 0)\|_2^2\,ds$. Therefore, the stochastic convolution integral

$$V(t) = \int_0^t U(t,s)g(s,0)\,dW_s$$

is adapted and continuous in t. By Theorem 3.2, we have

$$E\left\{\sup_{0\leqslant\rho\leqslant t}\left\|\int_0^\rho U(t,s)g(s,0)\,dW_s\right\|^p\right\} \leqslant e^{\gamma t}\frac{4C_p}{p}\int_0^t E\big\|g(s,0)\big\|_2^p\,ds < \infty. \tag{4.7}$$

Consider the integral equation

$$X_1(t) = \int_0^t U(t,s)f(s, X_{1s})\,ds + \int_0^t U(t,s)g(s,0)\,dW_s.$$

Theorem 5.6 and Corollary 5.8 of [17] provide us with a continuous adapted solution X_1 for this integral equation such that

$$\big(X_1^*(t)\big)^2 \leqslant 2Ce^{(2M+1)t}t\big\{1 + \big(V^*(t)\big)^2\big\} + 2\big(V^*(t)\big)^2,$$

where C is the constant appeared in Hypothesis 2.6(d). Therefore,

$$\big(X_1^*(t)\big)^p \leqslant 2^{1+p}C^{\frac{p}{2}}e^{\frac{p}{2}(2M+1)t}t^{\frac{p}{2}}\big\{1 + \big(V^*(t)\big)^p\big\} + 2^{1+\frac{p}{2}}\big(V^*(t)\big)^p, \quad \forall t \in [0, T].$$

Since by (4.7), $E(V^*(t))^p < \infty$, we obtain that $E(X_1^*(t))^p < \infty$ for $t \in [0, T]$. We proceed by induction and assuming that X_n has been defined and $E(X_n^*(t))^p < \infty$, we consider the integral equation

$$X_{n+1}(t) = \int_0^t U(t,s)f(s, X_{n+1s})\,ds + \int_0^t U(t,s)g(s, X_{ns})\,dW_s. \tag{4.8}$$

Note that by Hypothesis 2.6(e),

$$\big\|g(s, X_{ns})\big\|_2 \leqslant C\sup_{0\leqslant\rho\leqslant s}\big\|X_n(\rho)\big\| + \big\|g(s,0)\big\|_2,$$

which yields

$$E\|g(s,X_{ns})\|_2^p \leqslant 2^p C^p E\big(X_n^*(s)\big)^p + 2^p E\|g(s,0)\|_2^p.$$

Therefore, the stochastic convolution integral $V_n(t) = \int_0^t U(t,s)g(s,X_{ns})\,dW_s$ is continuous and adapted in t and by Theorem 3.2,

$$E\left\{\sup_{0\leqslant\rho\leqslant t}\left\|\int_0^\rho U(t,s)g(s,X_{ns})\,dW_s\right\|^p\right\} \leqslant e^{\gamma t}\frac{4C_p}{p}\int_0^t E\|g(s,X_{ns})\|_2^p\,ds < \infty.$$

Hence, again by Theorem 5.6 and Corollary 5.8 of [17], we can find a unique continuous adapted solution X_{n+1} for (4.8) such that

$$\big(X_{n+1}^*(t)\big)^p \leqslant 2^{1+p}C^{\frac{p}{2}}e^{\frac{p}{2}(2M+1)t}t^{\frac{p}{2}}\big\{1+\big(V_n^*(t)\big)^p\big\} + 2^{1+\frac{p}{2}}\big(V_n^*(t)\big)^p, \quad \forall t\in[0,T]$$

and this immediately implies $E(X_{n+1}^*(t))^p < \infty$ for $t\in[0,T]$. Next, we are going to prove the convergence of the approximating solutions $\{X_n\}$ to a mild solution of (2.2). First, we make the difference of two consecutive terms:

$$\begin{aligned}X_{n+1}(t) - X_n(t) = {} & \int_0^t U(t,s)\big(f(s,X_{n+1s}) - f(s,X_{ns})\big)\,ds \\ & + \int_0^t U(t,s)\big(g(s,X_{ns}) - g(s,X_{n-1s})\big)\,dW_s.\end{aligned}$$

Using the Itô-type inequality (Theorem 2.7), we obtain that

$$\begin{aligned}&\|X_{n+1}(t) - X_n(t)\|^p \\ &\quad\leqslant p\int_0^t \|X_{n+1}(s) - X_n(s)\|^{p-2}\big\langle X_{n+1}(s) - X_n(s), f(s,X_{n+1s}) - f(s,X_{ns})\big\rangle\,ds \\ &\qquad + p\int_0^t \|X_{n+1}(s) - X_n(s)\|^{p-2}\big\langle X_{n+1}(s) - X_n(s), \big(g(s,X_{ns}) - g(s,X_{n-1s})\big)\,dW_s\big\rangle \\ &\qquad + \frac{p(p-1)}{2}\int_0^t \|X_{n+1}(s) - X_n(s)\|^{p-2}\|g(s,X_{ns}) - g(s,X_{n-1s})\|_2^2\,ds \\ &\quad= J_1(t) + J_2(t) + J_3(t).\end{aligned} \tag{4.9}$$

We estimate the mathematical expectation of each term on the right-hand side of (4.9). Hypothesis 2.6(c) ends immediately the task for the first integral:

$$J_1(t) \leqslant pM\int_0^t \|X_{n+1}(s) - X_n(s)\|^p\,ds. \tag{4.10}$$

Moreover, by Lemma 3.1 with $K = 3p$,

$$E\big(J_2^*(t)\big) \leqslant \frac{1}{2} E\Big(\sup_{0\leqslant s\leqslant t} \big\|X_{n+1}(s) - X_n(s)\big\|^p\Big) + \frac{(3p)^2}{2} \int_0^t E\big\|X_{n+1}(s) - X_n(s)\big\|^{p-2} \big\|g(s, X_{ns}) - g(s, X_{n-1s})\big\|_2^2 \, ds. \tag{4.11}$$

Applying the elementary inequality

$$u^{1-\alpha} v^{\alpha} \leqslant (1-\alpha)u + \alpha v,$$

which is true for $u, v \geqslant 0$ and all $0 \leqslant \alpha \leqslant 1$ to (4.11), we conclude that

$$E\big(J_2^*(t)\big) \leqslant \frac{1}{2} E\Big(\sup_{0\leqslant s\leqslant t} \big\|X_{n+1}(s) - X_n(s)\big\|^p\Big) + \frac{(3p)^2}{2}\left(1 - \frac{2}{p}\right) \int_0^t E\big\|X_{n+1}(s) - X_n(s)\big\|^p \, ds + 9C^2 p \int_0^t E\Big(\sup_{0\leqslant \rho\leqslant s} \big\|X_n(\rho) - X_{n-1}(\rho)\big\|^p\Big) \, ds. \tag{4.12}$$

By Hypothesis 2.6(e) and a similar argument as above for the third integral, we have

$$E\big(J_3(t)\big) \leqslant C^2 \frac{p(p-1)}{2} \left[\left(1 - \frac{2}{p}\right) \int_0^t E\big\|X_{n+1}(s) - X_n(s)\big\|^p \, ds + \frac{2}{p} \int_0^t E\Big(\sup_{0\leqslant \rho\leqslant s} \big\|X_n(\rho) - X_{n-1}(\rho)\big\|^p\Big) \, ds\right]. \tag{4.13}$$

Define for any positive integer $n \geqslant 2$,

$$\varphi_n(t) = \sup_{0\leqslant s\leqslant t} \big\|X_n(s) - X_{n-1}(s)\big\|^p, \quad t \in [0, T].$$

Substituting (4.10), (4.12) and (4.13) in the right side of (4.9), yields

$$E\big(\varphi_n(t)\big) \leqslant \beta_1 \int_0^t E\big(\varphi_n(s)\big) \, ds + \beta_2 \int_0^t E\big(\varphi_{n-1}(s)\big) \, ds,$$

in which

$$\beta_1 = 2pM + \big((3p)^2 + C^2 p(p-1)\big)\left(1 - \frac{2}{p}\right),$$

and

$$\beta_2 = C^2\left(2(p-1) + (3p)^2\frac{2}{p}\right).$$

Now, by virtue of the Gronwall inequality,

$$E\big(\varphi_n(t)\big) \leqslant \beta_2 e^{\beta_1 t}\int_0^t E\big(\varphi_{n-1}(s)\big)\,ds.$$

Let $D > 0$ be a constant such that $\varphi_1(t) \leqslant D$. Then an easy induction indicates that

$$E\big(\varphi_n(t)\big) \leqslant D\frac{(\beta_2 e^{\beta_1 T} t)^{n-1}}{(n-1)!}, \quad n = 2, 3, \ldots.$$

Therefore, $\{X_n\}$ is a Cauchy sequence in the space $L^p(\Omega, C(-r, T; H))$ and hence there exists a continuous adapted process $X(t)$ with $E(X^*(t))^p < \infty$ such that $E(\sup_{0\leqslant s\leqslant t}\|X_n(s) - X(s)\|^p) \to 0$. We show that $X(t)$ is the mild solution of (2.2). To this end, set

$$R_n(t) = X_{n+1}(t) - \int_0^t U(t,s)f(s, X_{n+1s})\,ds - \int_0^t U(t,s)g(s, X_{ns})\,dW_s,$$

and

$$R(t) = X(t) - \int_0^t U(t,s)f(s, X_s)\,ds - \int_0^t U(t,s)g(s, X_s)\,dW_s.$$

Of course $R_n(t) = 0$ for all $t \in [0, T]$. It is enough to show that given any $t \in [0, T]$ and arbitrary $v \in H$, we have $\langle v, R(t)\rangle = 0$ with probability one, then letting t ranges over all rational numbers and using the continuity of $R(t)$ and separability of the space H, we conclude that $R(t) = 0$ a.s. for all $t \in [0, T]$. First, we can extract a subsequence which we denote it by the same symbol $\{X_n\}$ such that $\sup_{0\leqslant s\leqslant t}\|X_n(s) - X(s)\| \to 0$, a.s. and hence,

$$\big\langle v, X_{n+1}(t)\big\rangle \to \big\langle v, X(t)\big\rangle, \quad \text{a.s.}$$

Next,

$$\left\langle v, \int_0^t U(t,s)f(s, X_{n+1s})\,ds\right\rangle = \int_0^t \big\langle U^*(t,s)v, f(s, X_{n+1s})\big\rangle\,ds,$$

where the star notation means conjugate operator. Since $X_{ns} \to X_s$ in C_H for all $s \in [0, t]$, demicontinuity of f implies that

$$\big\langle U^*(t,s)v, f(s, X_{n+1s})\big\rangle \to \big\langle U^*(t,s)v, f(s, X_s)\big\rangle, \quad \text{as } n \to \infty.$$

On the other hand, by Hypothesis 2.6(d) and the fact that $\|U^*(t,s)\| \leqslant 1$ for $s \in [0,t]$, we may write

$$\left|\left\langle U^*(t,s)v, f(s, X_{n+1s})\right\rangle\right| \leqslant C\|v\|\Big(2 + \sup_{0\leqslant\rho\leqslant s}\big\|X(\rho)\big\|\Big),$$

for large enough n. So, by Lebesgue's dominated convergence theorem,

$$\left\langle v, \int_0^t U(t,s) f(s, X_{n+1s})\,ds\right\rangle \to \left\langle v, \int_0^t U(t,s) f(s, X_s)\,ds\right\rangle, \quad \text{as } n \to \infty.$$

At last, for the convergence of the second integrals we can see by Theorem 3.2 that

$$\begin{aligned} E\left\{\left\|\int_0^t U(t,s)\big(g(s, X_{ns}) - g(s, X_s)\big)\,dW_s\right\|^p\right\} &\leqslant e^{\gamma t}\frac{4C_p}{p}\int_0^t E\big\|g(s, X_{ns}) - g(s, X_s)\big\|_2^p\,ds \\ &\leqslant C^p e^{\gamma t}\frac{4C_p}{p} t E\Big(\sup_{0\leqslant s\leqslant t}\big\|X_n(s) - X(s)\big\|^p\Big) \to 0. \end{aligned}$$

Therefore, we can find out another subsequence $\{X_{n'}\}$ such that

$$\left\langle v, \int_0^t U(t,s) g(s, X_{n's})\,dW_s\right\rangle \to \left\langle v, \int_0^t U(t,s) g(s, X_s)\,dW_s\right\rangle, \quad \text{as } n' \to \infty.$$

From the above convergence results, we obtain

$$\big\langle v, R(t)\big\rangle = \lim_{n'\to\infty}\big\langle v, R_{n'}(t)\big\rangle = 0.$$

This finishes the proof. □

5. Asymptotic behaviour of sample paths

In this section, we turn our attention to stability problem for the mild solutions of (2.2). Note that in order to know whether a mild solution is stable or not, we need, at the first place, to have the solution be defined on the whole time interval $[0,\infty)$. For this reason, we assume in this section that all hypotheses ensuring the existence of the mild solution hold on $[0,\infty)$ such that for each $T > 0$ the mild solution exists on $[0,T]$ and an argument based on the uniqueness result, enables us to define a unique mild solution on $[0,\infty)$. Two kinds of exponential stability to which the present section is devoted are stability in the p-th moment, $p \geqslant 2$, and stability of a.s. sample paths. For the sake of brevity, sometimes we denote by X^ϕ or similar notations the unique mild solution of (2.2) with the initial data ϕ.

Definition 5.1. Let $p \geqslant 2$. The mild solution X^ϕ of (2.2) is said to be exponentially asymptotically stable in the p-th moment if there exist constants $\alpha, \beta > 0$ such that for any mild solution Y^ψ of (2.2),

$$E\big\|X^\phi(t) - Y^\psi(t)\big\|^p \leqslant \alpha e^{-\beta t} E\|\phi - \psi\|_{C_H}^p, \quad \forall t \geqslant 0.$$

Now, we state our main result of this section on the stability in the p-th moment.

Theorem 5.2. *Let $p \geqslant 2$ and $\lambda > 0$. Assume that A, U, f and g satisfy Hypothesis 2.6 with parameter $-\lambda$. Let X^ϕ and Y^ψ be two mild solutions of (2.2) with the initial data ϕ and ψ, respectively. Then there exist constants $\alpha, a > 0$ such that*

$$E\Big(\sup_{0\leqslant\rho\leqslant t} e^{p\lambda\rho}\big\|X^\phi(\rho)-Y^\psi(\rho)\big\|^p\Big)\leqslant \alpha e^{at}E\|\phi-\psi\|_{C_H}^p,$$

for all $t \geqslant 0$.

Proof. Since by assumption, $X = X^\phi$ and $Y = Y^\psi$ are solutions of the integral equation (2.3), we have

$$X(t)=U(t,0)\phi(0)+\int_0^t U(t,s)f(s,X_s)\,ds+\int_0^t U(t,s)g(s,X_s)\,dW_s, \tag{5.1}$$

and

$$Y(t)=U(t,0)\psi(0)+\int_0^t U(t,s)f(s,Y_s)\,ds+\int_0^t U(t,s)g(s,Y_s)\,dW_s. \tag{5.2}$$

Note that replacing the parameter $\lambda \in \mathbb{R}$ in Hypothesis 2.1 with $-\lambda$ where $\lambda > 0$ implies that the evolution family $U(t,s)$ is exponentially stable; i.e., $\|U(t,s)\| \leqslant e^{-\lambda(t-s)}$ for all $0 \leqslant s \leqslant t$. Subtracting (5.2) from (5.1) and then using the Itô-type inequality, yield

$$\begin{aligned}
\big\|X(t)-Y(t)\big\|^p &\leqslant e^{-p\lambda t}\big\|\phi(0)-\psi(0)\big\|^p\\
&\quad+p\int_0^t e^{-p\lambda(t-s)}\big\|X(s)-Y(s)\big\|^{p-2}\big\langle X(s)-Y(s), f(s,X_s)-f(s,Y_s)\big\rangle\,ds\\
&\quad+p\int_0^t e^{-p\lambda(t-s)}\big\|X(s)-Y(s)\big\|^{p-2}\big\langle X(s)-Y(s), \big(g(s,X_s)-g(s,Y_s)\big)\,dW_s\big\rangle\\
&\quad+\frac{p(p-1)}{2}\int_0^t e^{-p\lambda(t-s)}\big\|X(s)-Y(s)\big\|^{p-2}\big\|g(s,X_s)-g(s,Y_s)\big\|_2^2\,ds.
\end{aligned} \tag{5.3}$$

The special kind of monotone condition we have imposed on the nonlinearity f helps us to estimate the first integral on the right side of (5.3). In fact, if X is a stochastic process defined on the time interval $-r \leqslant t < \infty$, then with the notations of Section 2 we have for all $s \geqslant 0$ that $X(s) = X_s(0)$. Therefore, multiplying both sides of (5.3) by $e^{p\lambda t}$ and using Hypothesis 2.6(c) and (e), we get

$$\sup_{0\leqslant\rho\leqslant t} e^{p\lambda\rho}\|X(\rho)-Y(\rho)\|^p$$
$$\leqslant \|\phi(0)-\psi(0)\|^p + \left(pM + \frac{p(p-1)}{2}C^2\right)\int_0^t e^{p\lambda s}\|X(s)-Y(s)\|^p\,ds$$
$$+ p\sup_{0\leqslant\rho\leqslant t}\left|\int_0^\rho e^{p\lambda s}\|X(s)-Y(s)\|^{p-2}\big\langle X(s)-Y(s),\big(g(s,X_s)-g(s,Y_s)\big)\,dW_s\big\rangle\right|. \tag{5.4}$$

Let

$$M(t)=\int_0^t \big(g(s,X_s)-g(s,Y_s)\big)\,dW_s.$$

Then $M(t)$ is an H-valued continuous martingale with quadratic variation

$$[M](t)=\int_0^t \|g(s,X_s)-g(s,Y_s)\|_2^2\,ds.$$

Therefore, by Lemma 3.1 with $K=3p$,

$$E\left(\sup_{0\leqslant\rho\leqslant t}\left|\int_0^\rho e^{p\lambda s}\|X(s)-Y(s)\|^{p-2}\big\langle X(s)-Y(s),dM(s)\big\rangle\right|\right) \tag{5.5}$$
$$\leqslant \frac{1}{2p}E\Big(\sup_{0\leqslant\rho\leqslant t} e^{p\lambda\rho}\|X(\rho)-Y(\rho)\|^p\Big)+\frac{p(3C)^2}{2}\int_0^t e^{p\lambda s}E\|X(s)-Y(s)\|^p\,ds.$$

Now, define $\Lambda(t)=\sup_{0\leqslant\rho\leqslant t} e^{p\lambda\rho}\|X(\rho)-Y(\rho)\|^p$ for all $t\geqslant 0$. We take the mathematical expectation of both sides of (5.4) and apply (5.5) to conclude that

$$\frac{1}{2}E\big(\Lambda(t)\big)\leqslant E\|\phi(0)-\psi(0)\|^p+\left(pM+\frac{p(p-1)}{2}C^2+\frac{(3pC)^2}{2}\right)\int_0^t E\big(\Lambda(s)\big)\,ds.$$

Thus, by the Gronwall inequality,

$$E\big(\Lambda(t)\big)\leqslant 2e^{at}E\|\phi(0)-\psi(0)\|^p\leqslant 2e^{at}E\|\phi-\psi\|_{C_H}^p,\quad \forall t\geqslant 0,$$

in which $a=2pM+p(p-1)C^2+(3pC)^2$. □

Corollary 5.3. *Suppose that all conditions of Theorem* 5.2 *hold. Then for any mild solutions X^ϕ and Y^ψ for* (2.2)*, we have*

$$E\|X^\phi(t)-Y^\psi(t)\|^p\leqslant 2e^{-(p\lambda-a)t}E\|\phi-\psi\|_{C_H}^p,\quad \forall t\geqslant 0,$$

where $a = 2pM + p(p-1)C^2 + (3pC)^2$. Consequently, if $p\lambda > a$, then the mild solution X^ϕ is exponentially asymptotically stable in the p-th moment.

Corollary 5.4. *Assume that all conditions of Theorem* 5.2 *hold. If $f(t,0) = g(t,0) = 0$ for all $t \in [0,\infty)$ and if $p\lambda > 2pM + p(p-1)C^2 + (3pC)^2$, then the trivial solution of* (2.2) *is exponentially asymptotically stable in the p-th moment.*

It is worth pointing out that the results derived in [5,16] are somewhat restrictive for many practical applications. In fact, the situation turns out to be rather complicated when one considers the general functional differential equations. Thus, our main results, Theorem 5.2 and Corollary 5.3, improve some of those from [5,16,26].

Since by Theorem 4.3, the mild solutions of (2.2) have continuous sample paths, it is reasonable to consider the stability of the sample paths. To this end, we will follow the same method as in [14,16].

Theorem 5.5. *Suppose that all hypotheses of Theorem* 5.2 *hold and the constant $a = 2pM + p(p-1)C^2 + (3pC)^2$ be the same as in Theorem* 5.2*. If $p\lambda > a$, then for any two mild solutions X^ϕ and Y^ψ for* (2.2) *with the initial data ϕ and ψ, respectively*

$$\limsup_{t\to\infty}(1/t)\log\left\|X^\phi(t) - Y^\psi(t)\right\| \leqslant -(p\lambda - a)/2p, \quad \text{w.p. } 1.$$

Proof. Let n be an arbitrary natural number. By assumption, $X = X^\phi$ and $Y = Y^\psi$ satisfy the integral equation (2.3); so we have

$$X(t) = U(t,n)X(n) + \int_n^t U(t,s)f(s,X_s)\,ds + \int_n^t U(t,s)g(s,X_s)\,dW_s,$$

and

$$Y(t) = U(t,n)Y(n) + \int_n^t U(t,s)f(s,Y_s)\,ds + \int_n^t U(t,s)g(s,Y_s)\,dW_s,$$

for all $n \leqslant t \leqslant n+1$. Subtracting the above two equations and using the Itô-type inequality, we obtain

$$\begin{aligned}
\left\|X(t) - Y(t)\right\|^p \leqslant{}& e^{-p\lambda(t-n)}\left\|X(n) - Y(n)\right\|^p \\
&+ p\int_n^t e^{-p\lambda(t-s)}\left\|X(s) - Y(s)\right\|^{p-2}\big\langle X(s) - Y(s), f(s,X_s) - f(s,Y_s)\big\rangle\,ds \\
&+ p\int_n^t e^{-p\lambda(t-s)}\left\|X(s) - Y(s)\right\|^{p-2}\big\langle X(s) - Y(s), \big(g(s,X_s) - g(s,Y_s)\big)\,dW_s\big\rangle \\
&+ \frac{p(p-1)}{2}\int_n^t e^{-p\lambda(t-s)}\left\|X(s) - Y(s)\right\|^{p-2}\left\|g(s,X_s) - g(s,Y_s)\right\|_2^2\,ds, \qquad (5.6)
\end{aligned}$$

for each $n \leqslant t \leqslant n+1$. Fix an $\epsilon_n > 0$ and denote by $J_k(t)$, $k = 1,2,3$, the k-th integral in the right side of (5.6). Therefore,

$$
\begin{aligned}
P\Big\{\sup_{n\leqslant t\leqslant n+1}\big\|X(t)-Y(t)\big\|^p>\epsilon_n\Big\} &\leqslant P\Big\{\sup_{n\leqslant t\leqslant n+1}e^{-p\lambda(t-n)}\big\|X(n)-Y(n)\big\|^p>\frac{\epsilon_n}{4}\Big\}\\
&\quad+P\Big\{\sup_{n\leqslant t\leqslant n+1}p\big|J_1(t)\big|>\frac{\epsilon_n}{4}\Big\}\\
&\quad+P\Big\{\sup_{n\leqslant t\leqslant n+1}p\big|J_2(t)\big|>\frac{\epsilon_n}{4}\Big\}\\
&\quad+P\Big\{\sup_{n\leqslant t\leqslant n+1}\frac{p(p-1)}{2}J_3(t)>\frac{\epsilon_n}{4}\Big\}\\
&=A_1+A_2+A_3+A_4.
\end{aligned}
$$

Now, we obtain an estimate depending on $n,\epsilon_n>0$ for each term A_i, $i=1,2,3,4$. By Theorem 5.2 and Markov inequality,

$$
\begin{aligned}
A_1 &\leqslant (4/\epsilon_n)E\big\|X(n)-Y(n)\big\|^p\\
&\leqslant (8/\epsilon_n)e^{-(p\lambda-a)n}E\big\|\phi-\psi\big\|_{C_H}^p.
\end{aligned}
\tag{5.7}
$$

Next, since by Hypothesis 2.6(c) f is semimonotone with parameter M, we have

$$
A_2\leqslant (4pM/\epsilon_n)E\Bigg(\sup_{n\leqslant t\leqslant n+1}\int_n^t e^{-p\lambda(t-s)}\big\|X(s)-Y(s)\big\|^p\,ds\Bigg),
$$

and again by Theorem 5.2,

$$
\begin{aligned}
A_2 &\leqslant (4pM/\epsilon_n)\int_n^{n+1}E\big\|X(s)-Y(s)\big\|^p\,ds\\
&\leqslant (8pM/\epsilon_n)e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p.
\end{aligned}
\tag{5.8}
$$

Furthermore, by Markov inequality, Hypothesis 2.6(e) and Lemma 3.1 with $K=3e^a$, we conclude that

$$
\begin{aligned}
A_3 &\leqslant (4p/\epsilon_n)\big[2e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p+\big(3Ce^a\big)^2e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p\big]\\
&=(4pL/\epsilon_n)e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p,
\end{aligned}
\tag{5.9}
$$

where $L=2+(3Ce^a)^2$. Finally, with the same reasoning as above and since g satisfies Lipschitz condition with constant $C>0$,

$$
A_4\leqslant \big(4C^2p(p-1)/\epsilon_n\big)e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p.
\tag{5.10}
$$

From (5.7), (5.8), (5.9) and (5.10) one can conclude that there exists a positive constant γ such that for any $\epsilon_n>0$,

$$
P\Big\{\sup_{n\leqslant t\leqslant n+1}\big\|X(t)-Y(t)\big\|^p>\epsilon_n\Big\}\leqslant(\gamma/\epsilon_n)e^{-(p\lambda-a)n}E\|\phi-\psi\|_{C_H}^p.
$$

Choose $\epsilon_n = e^{-(p\lambda-a)n/2} E\|\phi-\psi\|_{C_H}^p$. Then since the series $\sum_{n=1}^\infty e^{-(p\lambda-a)n/2}$ is convergent, the Borel–Cantelli's lemma provides us with a finite positive random variable T and a set $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0)=1$ such that for any $\omega \in \Omega_0$ and all $t > T(\omega)$,

$$\big\|X(t)-Y(t)\big\|^p \leqslant \beta e^{-(p\lambda-a)t/2} E\|\phi-\psi\|_{C_H}^p,$$

where $\beta = e^{(p\lambda-a)/2}$. This completes the proof of the theorem. $\square$

Corollary 5.6. *Assume that all conditions of Theorem* 5.2 *hold and* $f(t,0)=g(t,0)=0$. *If* $p\lambda > a$, *then for any mild solution* X^ϕ *of Eq.* (2.2) *with initial data* $\phi \in C_{\mathcal{F}_0}^p$, *we have*

$$\limsup_{t\to\infty}(1/t)\log\big\|X(t)\big\| \leqslant -(p\lambda-a)/2p, \quad \textit{w.p. } 1;$$

that is, the sample path of the trivial solution of (2.2) *is exponentially asymptotically stable.*

6. Applications

The stability theory of semilinear stochastic heat equations has been well-studied by several authors among them we may point out [12,13,25] in the case of Lipschitz nonlinearity and [14] in the case of monotone one. In this section, we study the stochastic functional parabolic initial-boundary value problems of monotone-type and by virtue of the results derived in the previous sections, we justify the existence and stability of the mild solutions. Many problems such as stochastic functional and delay heat equations can be treated in these settings.

Example 1. Let $B_t, t \in [0,\infty)$, be the real standard Brownian motion and consider the stochastic semilinear delay heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \alpha_1 \int_{-r_1}^{0} F\big(x, u(t+s,x)\big)\,ds + \alpha_2 \int_{-r_2}^{0} G\big(x, u(t+s,x)\big)\,ds\,\dot{B}_t, \quad r_1, r_2, \alpha_1, \alpha_2 \geqslant 0, \tag{6.1}$$

with the boundary conditions $u(t,0)=u(t,1)=0$, $t \geqslant 0$, and initial condition $u(\theta,x)=\phi(\theta,x)$ for all $\theta \in [-r,0]$, $x \in [0,1]$, where

$$\phi(\theta,.) \in L^2(0,1) \quad \text{for } \theta \in [-r,0],\ r \geqslant r_1, r_2; \qquad \phi(.,x) \in C_{\mathbb{R}} \quad \text{for } x \in [0,1].$$

We formulate the problem in such a way that the theory developed in the previous sections can be applied.

Hypothesis 6.1.

(a) $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition;
(b) There exist a function $a \in L^2(0,1)$ and a constant $C > 0$ such that

$$\big|F(x,y)\big| \leqslant a(x) + C|y| \quad \text{and} \quad \big|G(x,y)\big| \leqslant a(x) + C|y|, \quad \forall x \in [0,1],\ \forall y \in \mathbb{R};$$

(c) For all $\xi_1, \xi_2 \in C(-r_1, 0; \mathbb{R})$ we have

$$\int_{-r_1}^{0} \bigl\{F\bigl(x, \xi_1(s)\bigr) - F\bigl(x, \xi_2(s)\bigr)\bigr\}\bigl(\xi_1(0) - \xi_2(0)\bigr)\, ds \leqslant 0, \quad \forall x \in [0,1];$$

(d) $G(x, .)$ is Lipschitz continuous uniformly with respect to x with the Lipschitz constant $C > 0$:

$$\bigl|G(x, y_1) - G(x, y_2)\bigr| \leqslant C|y_1 - y_2|, \quad \forall x \in [0,1];$$

(e) $\phi : [-r, 0] \to L^2(0,1)$ is a continuous process satisfying

$$E\left(\sup_{-r\leqslant\theta\leqslant 0} \int_0^1 \bigl|\phi(\theta, x)\bigr|^2\, dx\right) < \infty;$$

that is, $\phi \in C^2_{\mathcal{F}_0}$.

Let $H = L^2(0,1)$ and $A = \frac{\partial^2}{\partial x^2}$ be the linear operator of second order differentiation with respect to spatial variable with the domain

$$\mathcal{D}(A) = \bigl\{u \in L^2(0,1)\colon\ u', u'' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\bigr\}.$$

Then $A : \mathcal{D}(A) \subseteq H \to H$ generates a strongly continuous semigroup $S(t)$ satisfying $\|S(t)\| \leqslant e^{-\pi^2 t}$ for all $t \geqslant 0$. If we define $f, g : C_H \to H$ by

$$f(\psi)(x) = \alpha_1 \int_{-r_1}^{0} F\bigl(x, \psi(s)(x)\bigr)\, ds, \qquad g(\psi)(x) = \alpha_2 \int_{-r_2}^{0} G\bigl(x, \psi(s)(x)\bigr)\, ds,$$

then the initial-boundary value problem (6.1) can be written as an abstract functional evolution equation:

$$du = \bigl[Au(t) + f(u_t)\bigr]\, dt + g(u_t)\, dB_t, \quad u_0 = \phi. \tag{6.2}$$

By Hypothesis 6.1(b), we get that

$$\begin{aligned}
\bigl\|f(\psi)\bigr\|^2 &\leqslant \alpha_1^2 r_1^2 \int_0^1 \int_{-r_1}^{0} \bigl|F\bigl(x, \psi(s)(x)\bigr)\bigr|^2\, ds\, dx \\
&\leqslant 2\alpha_1^2 r_1^2 \int_0^1 \int_{-r_1}^{0} \bigl\{a^2(x) + \bigl|\psi(s)(x)\bigr|^2\bigr\}\, ds\, dx \\
&\leqslant 2\alpha_1^2 r^3 \Bigl\{\|a\|^2_{L^2} + \sup_{-r\leqslant s\leqslant 0} \bigl\|\psi(s)\bigr\|^2_{L^2}\Bigr\}.
\end{aligned}$$

Therefore, f is continuous and one can find a constant $C > 0$ such that

$$\|f(\psi)\| \leqslant C\big(1 + \|\psi\|_{C_H}\big), \quad \forall \psi \in C_H.$$

On the other hand, Hypothesis 6.1(c) gives us the monotonicity of f: given any $\psi_1, \psi_2 \in C_H$, we have

$$\begin{aligned}
&\big\langle f(\psi_1) - f(\psi_2), \psi_1(0) - \psi_2(0)\big\rangle \\
&\quad = \int_0^1 \int_{-r_1}^0 \big[F\big(x, \psi_1(s)(x)\big) - F\big(s, \psi_2(s)(x)\big)\big]\big[\psi_1(0)(x) - \psi_2(0)(x)\big]\, ds\, dx \leqslant 0.
\end{aligned}$$

Moreover, by Hypothesis 6.1(d), we can conclude that

$$\begin{aligned}
\big\|g(\psi_1) - g(\psi_2)\big\|^2 &\leqslant \alpha_2^2 r_2^2 \int_0^1 \int_{-r_2}^0 \big|G\big(x, \psi_1(s)(x)\big) - G\big(x, \psi_2(s)(x)\big)\big|^2\, ds\, dx \\
&\leqslant C^2 \alpha_2^2 r_2^2 \int_0^1 \int_{-r_2}^0 \big|\psi_1(s)(x) - \psi_2(s)(x)\big|^2\, ds\, dx \\
&\leqslant C^2 \alpha_2^2 r^3 \sup_{-r \leqslant s \leqslant 0} \big\|\psi_1(s) - \psi_2(s)\big\|_{L^2}^2.
\end{aligned}$$

Therefore, the required Lipschitz property for g also holds. Now, Corollary 5.4 implies that if $F(x, 0) = G(x, 0) = 0$ for all $x \in [0, 1]$, then we have for any mild solution u^ϕ of (6.2),

$$E\big\|u^\phi(t)\big\|^p \leqslant 2e^{-(p\pi^2 - a)t}\|\phi\|_{C_H}^p, \quad \forall t \geqslant 0,$$

where $a = p(p-1)C\alpha_2 r^{3/2} + (3Cp\alpha_2)^2 r^3$. Thus, if $p\pi^2 > a$, then the trivial solution of (6.1) is exponentially asymptotically stable in p-th mean.

Example 2. Let α and μ be two positive real numbers and consider the semilinear stochastic heat equation with time lags $r_1, r_2 > 0$ of the type introduced in [6]:

$$du(t, x) = \left[\mu \frac{\partial^2}{\partial x^2} u(t, x) + \alpha \int_{-r_1}^0 u(t + s, x)h(s)\, ds\right] dt + k\big(u(t - r_2, x)\big)\, dB_t, \tag{6.3}$$

with the boundary conditions $u(t, 0) = u(t, \pi) = 0$, $t \geqslant 0$, and initial condition $u(\theta, x) = \phi(\theta, x)$ for all $\theta \in [-r, 0]$, $x \in [0, \pi]$, where

$$\phi(\theta, .) \in L^2(0, \pi) \quad \text{for } \theta \in [-r, 0],\ r \geqslant r_1, r_2; \qquad \phi(., x) \in C_{\mathbb{R}} \quad \text{for } x \in [0, \pi].$$

Also B_t is the real standard Brownian motion, $k : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $C > 0$ and $h : [-r_1, 0] \to \mathbb{R}$ is continuous. Let $H = L^2(0, \pi)$ and $A = \mu \frac{\partial^2}{\partial x^2}$ with the domain

$$\mathcal{D}(A) = \big\{u \in L^2(0, \pi) \colon u', u'' \in L^2(0, \pi) \text{ and } u(0) = u(\pi) = 0\big\}.$$

Then $A:\mathcal{D}(A)\subseteq H\to H$ generates a strongly continuous semigroup $S(t)$ satisfying $\|S(t)\|\leqslant e^{-\mu t}$ for all $t\geqslant 0$. Now, define $f,g:C_H\to H$ by

$$f(\psi)(x)=\alpha\int_{-r_1}^{0}\psi(s)(x)h(s)\,ds,\qquad g(\psi)(x)=k\big(\psi(-r_2)(x)\big).$$

It is easy to see that g is Lipschitz continuous and f satisfies a linear growth condition. On the other hand, it is clear that if we assume that

$$\int_{-r_1}^{0}h(s)\big(\xi_1(s)-\xi_2(s)\big)\big(\xi_1(0)-\xi_2(0)\big)\,ds\leqslant 0,\quad \forall\xi_1,\xi_2\in C(-r_1,0;\mathbb{R}),$$

then f is also monotone in the sense of Hypothesis 2.6(c). Now, Corollary 5.4 implies that if $k(0)=0$, then we have for any mild solution u^{ϕ} of (6.3),

$$E\big\|u^{\phi}(t)\big\|^{p}\leqslant 2e^{-(p\mu-a)t}\|\phi\|_{C_H}^{p},\quad \forall t\geqslant 0,$$

where $a=p(p-1)C+(3pC)^2$. Thus, if $2\leqslant p<\frac{\mu+C}{C+9C^2}$, then the trivial solution of (6.3) is p-th mean and almost sure exponentially asymptotically stable. In particular case $k(x)=\beta x,\beta>0$, and $h(x)=1$, Taniguchi [26] obtained the required condition for the mean-square stability of (6.3) as $\mu>3(\frac{(\alpha r_1)^2}{\mu}+\beta^2)e^{\mu r}$, while according to the results of this paper, the same type of stability is guaranteed when $\mu>\beta+18\beta^2$ which is obviously better and does not depend on the time lag $r_1>0$.

Example 3. Let D be a bounded domain (that is an open connected set) in $\mathbb{R}^n$ with smooth boundary ∂D. Consider the stochastic functional initial-boundary value problem

$$\begin{cases}\dfrac{\partial u}{\partial t}=A(t,x)u(t,x)+f\big(t,x,u_t(x)\big)+g\big(t,x,u_t(x)\big)\dfrac{\partial W(t,x)}{\partial t},\\ u(t,x)=\phi(t,x),\quad \text{on } [-r,0]\times D,\\ u(t,x)=0,\quad \text{on } [0,\infty)\times\partial D,\end{cases}\tag{6.4}$$

where $W(t,x)$ is a space–time Brownian motion with the self-adjoint, positive and nuclear covariance operator Q and

$$A(t,x)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{n}b_i(t,x)\frac{\partial}{\partial x_i}+c(t,x),$$

is a uniformly elliptic operator; i.e., there exists a constant $\delta>0$ such that

$$\sum_{i,j=1}^{n}a_{ij}(t,x)\xi_i\xi_j\geqslant\delta\sum_{i=1}^{n}\xi_i^2,$$

for each $t\in[0,\infty)$, a.e. $x\in D$ and for all real vectors $\xi\in\mathbb{R}^n$. Assume that the coefficients $a_{ij}=a_{ji},b_i$ and c, $i,j=1,2,\dots,n$, are real-valued smooth functions on $[0,\infty)\times\bar{D}$ with bounded derivatives. Moreover, there exist positive constants β_1 and β_2 such that $|b_i(t,x)|\leqslant\beta_1$ and $c(t,x)\leqslant-\beta_2$ for all $t\in[0,\infty)$ and a.e. $x\in D$. To bring our problem in a situation that the stability results of this paper can be applied, we should impose the following further hypotheses on $f,g:[0,\infty)\times D\times C_{\mathbb{R}}\to\mathbb{R}$.

Hypothesis 6.2.

(a) There exist a function $a \in L^2(D)$ and a constant $C > 0$ such that

$$\big|f(t,x,\psi)\big| \leqslant a(x) + C\|\psi\|_{C_{\mathbb{R}}},$$
$$\big|g(t,x,\psi)\big| \leqslant a(x) + C\|\psi\|_{C_{\mathbb{R}}},$$

for all $t \in [0,\infty)$, $x \in D$ and $\psi \in C_{\mathbb{R}}$;

(b) $g(t,x,.)$ is uniformly Lipschitz continuous with constant $C > 0$; i.e.,

$$\big|g(t,x,\psi_1) - g(t,x,\psi_2)\big| \leqslant C\|\psi_1 - \psi_2\|_{C_{\mathbb{R}}},$$

for all $t \in [0,\infty)$, $x \in D$ and $\psi_1, \psi_2 \in C_{\mathbb{R}}$;

(c) $f(t,x,.)$ is semimonotone with parameter M uniformly with respect to $t \in [0,\infty)$ and $x \in D$; i.e.,

$$\big(f(t,x,\psi_1) - f(t,x,\psi_2)\big)\big(\psi_1(0) - \psi_2(0)\big) \leqslant M\big(\psi_1(0) - \psi_2(0)\big)^2,$$

for all $\psi_1, \psi_2 \in C_{\mathbb{R}}$;

(d) $\phi : [-r,0] \to L^2(D)$ is a continuous process satisfying

$$E\bigg(\sup_{-r\leqslant\theta\leqslant 0} \int_D \big|\phi(\theta,x)\big|^2\,dx\bigg) < \infty,$$

that is $\phi \in C^2_{\mathcal{F}_0}$.

Let $H = L^2(D)$ and for each $t \in [0,\infty)$ define the operator $A(t)$ on H by

$$\mathcal{D}\big(A(t)\big) = H^2(D) \cap H^1_0(D),$$

and $A(t)u = A(t,x)u(t,x)$ for all $u \in \mathcal{D}(A(t))$. Denote by $\bar{f}$ and $\bar{g}$ the functions defined on $[0,\infty) \times C_H$ by

$$\bar{f}(t,\psi)(x) = f\big(t,x,\psi(x)\big), \quad \psi \in C_H,$$

and

$$\big(\bar{g}(t,\psi)v\big)(x) = g\big(t,x,\psi(x)\big)v(x), \quad \psi \in C_H,\ v \in K,$$

respectively, where K is a Hilbert space continuously embedded into H. If $K_0 = Q^{\frac{1}{2}}K$, then the initial-boundary value problem (6.4) can be written as the stochastic functional evolution equation

$$dX(t) = \big[A(t)X(t) + \bar{f}(t,X_t)\big]\,dt + \bar{g}(t,X_t)\,d\tilde{W}_t, \quad X_0 = \phi, \tag{6.5}$$

in which $\bar{g} : [0,\infty) \times C_H \to L_2(K_0,H)$ and $\tilde{W}_t$ is a cylindrical Brownian motion on K_0. It is well known (see e.g. [24]) that if $\beta_2 > \frac{n\beta_1}{2\delta} + \frac{\delta}{2}$, then there exists $\lambda > 0$ such that the family $A(t), t \geqslant 0$, generates a strong evolution operator $U(t,s)$ which is exponentially bounded with parameter $-\lambda$ on $[0,\infty)$. In addition to this, $\bar{f}$ and $\bar{g}$ satisfy Hypothesis 2.6(a)–(e). Thus, if $2\lambda > a$, then any mild solution of (6.5) which is called a *generalized solution* of (6.4) with the initial data $\phi \in C^2_{\mathcal{F}_0}$ is exponentially asymptotically stable in the second moment. Moreover, by Corollary 5.6, if $f(t,x,0) = g(t,x,0) = 0$ for all $t \in [0,\infty)$ and $x \in D$, and if $2\lambda > a$, then for any mild solution $X(t)$ of (6.5), we have

$$\limsup_{t\to\infty}(1/t)\log\|X(t)\| \leqslant -(2\lambda - a)/4, \quad \text{w.p. } 1.$$

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